Monopoly, Non-linear Pricing, and Imperfect Information:
A Reconsideration of the Insurance Market

Dezsö Szalay

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Monopoly, Non-linear Pricing, and Imperfect Information: A Reconsideration of the Insurance Market*

Dezsö Szalay

University of Warwick

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Abstract

I reconsider Stiglitz's (1977) problem of monopolistic insurance with a continuum of types. Using a suitable transformation of control variables I obtain an analytical characterization of the optimal insurance policies. Closed form solutions and comparative statics results for special cases are provided.

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*Correspondence can be sent to the author at Department of Economics, University of Warwick, Gibbet Hill Road, CV4 7AL Coventry, UK or by email to dezso.szalay@warwick.ac.uk. I thank Martin Hellwig, Elu von Thadden, and Eric Toulemonde for helpful suggestions. Remaining errors are my own.

1 Introduction

In a seminal contribution Stiglitz (1977) asked how a monopolistic insurance company should sell insurance policies to consumers who know their demand for insurance while the monopolist does not. In particular, consumers know how likely they are to have an accident while the monopolist knows only the distribution of accident risks in the population. Stiglitz showed that the monopolist offers a menu of insurance contracts with different contracts tailored to the demand of consumers with different accident probabilities. In designing this menu of contracts, the monopolist must make sure that consumers with a relatively high demand for insurance have no incentive to understate their demand. As a result, the insurance contracts offered to low demand consumers are relatively unattractive. These consumers receive too little insurance relative to the socially optimal level of insurance. It is only the consumers with the very highest demand for insurance who receive the socially optimal level of insurance. Mussa and Rosen (1978) and Maskin and Riley (1984a) have shown that these insights extend in general to the theory of nonlinear pricing, and in fact even more generally to any sort of screening problem.[1]

But beyond these qualitative insights our current understanding of the insurance problem with accident probabilities drawn from a continuous distribution (i.e., when there is a continuum of types) is still limited. Stiglitz (1977) provides a partial description of the solution but no general description of the menu of optimal contracts. The reason is that consumers motive for demanding insurance in the first place, i.e., their risk aversion, complicates the monopolist's problem of contract design to the point where it becomes hardly tractable.

In this paper I provide the full solution of the problem. A suitable transformation of the space of control variables inspired by Grossman and Hart (1983) allows me to derive a formulation of the contracting problem that can be solved with much more ease than the original one. In particular, I have the monopolist offer utility contracts rather than the original insurance contracts. This simple change of variables renders the monopolist's profit function concave in the control variables, but at the same time it linearizes the incentive constraints. The solution takes the form of a pair of integral equations that can be solved for specific utility and density functions, in particular for a special case of a CRRA utility function and various distribution of types. The solution displays natural comparative statics properties: consumers receive better contracts when they are richer and when accident damages are smaller. In addition, consumers receive less insurance, in the sense

[1] See Laffont and Martimort (2002) for a recent and comprehensive survey of many problems that share the same economic trade-offs.

that the difference in their utility levels conditional on an accident occurring and conditional on no accident occurring is larger, when they are richer.

This paper adds to a small set of studies that have attacked problems of screening with risk averse agents and uncertainty. Salanié (1990) and Laffont and Rochet (1998) study the regulation of a risk averse firm. Matthews (1983) and Maskin and Riley (1984b) study auctions with risk averse buyers. In companion work I study incentive compatible risk sharing between two risk averse agents. The small size of this literature is not a sign that adverse selection is automatically unimportant as soon as agents are risk averse[2]. Rather it is a sign that risk aversion introduces significant technical difficulties. The contribution of the present analysis to this literature is a technical one: I provide a relatively simple way to analyze problems of screening with risk averse agents, when uncertainty is modeled as a two-outcome process. The method of analysis may prove useful in other applications of the two outcome model.

The remainder of this short paper is structured as follows: To develop the reader's intuition for my approach I introduce the model and begin my analysis in section 2 with the two type case. In section 3 I treat the case of a continuum of types. I have relegated the lengthy arguments in the proofs to the appendix.

2 The Two Type Case

There is a single insurance company and two groups of individuals. An individual in group 1 has an accident with probability equal to $\overline{\theta}$, and individual in group 2 has an accident with probability equal to $\underline{\theta}$, where $\overline{\theta} > \underline{\theta}$. An accident causes a monetary loss d. The proportion of the mass 1 population with low probability of accident (group 2) is λ. Individuals have concave von-Neumann-Morgenstern utility functions $u(w)$ defined over wealth w. An insurance contract is the right to the payment B conditional on having an accident at unconditional price α. Let $\beta = B - \alpha$ denote the net reimbursement after an accident, and let w_a and w_{na} denote the wealth conditional on an accident and no accident, respectively, when the individual has bought an insurance contract $\{\alpha, \beta\}$. Note that

$$\begin{aligned} w_a(\beta) &= w - d + \beta, \text{ and} \\ w_{na}(\alpha) &= w - \alpha. \end{aligned}$$

[2] There is a large literature on incentive compatible taxation starting with Mirrlees (1971) in which agents typically have concave utility functions. However, in contrast to the papers cited in the text, there is no uncertainty and hence no risk in the proper sense.

Thus, an individual's expected utility from buying insurance contract $\{\alpha, \beta\}$ is

$$U(\alpha, \beta, \theta) = \theta u(w_a(\beta)) + (1-\theta) u(w_{na}(\alpha)) \text{ for } \theta \in \{\underline{\theta}, \overline{\theta}\}.$$

The insurance company offers contracts $\{\underline{\alpha}, \underline{\beta}\}$ and $\{\overline{\alpha}, \overline{\beta}\}$ to insurants in order to maximize its profit

$$\pi = \lambda\left(-\underline{\theta}\underline{\beta} + (1-\underline{\theta})\underline{\alpha}\right) + (1-\lambda)\left(-\overline{\theta}\overline{\beta} + \left(1-\overline{\theta}\right)\overline{\alpha}\right).$$

The insurance company must offer contracts $\{\underline{\alpha}, \underline{\beta}\}$ and $\{\overline{\alpha}, \overline{\beta}\}$ such that an individual with a low probability of accident prefers to buy contract $\{\underline{\alpha}, \underline{\beta}\}$ rather than contract $\{\overline{\alpha}, \overline{\beta}\}$ or no contract at all. Likewise, a high risk individual must prefer to buy contract $\{\overline{\alpha}, \overline{\beta}\}$ rather than contract $\{\underline{\alpha}, \underline{\beta}\}$ or no contract at all.

These incentive and participation constraints are complex to analyze, because they are non-linear in the insurance contract. However, note that $U(\alpha, \beta, \theta)$ is linear in $u(w_a(\beta))$ and $u(w_{na}(\alpha))$. Therefore, I switch variables and let the insurance company offer utility contracts rather than contracts in the $\{\alpha, \beta\}$ space.

To ease notation let $\overline{u}_a \equiv \left(w_a\left(\overline{\beta}\right)\right), \underline{u}_{na} \equiv u(w_{na}(\underline{\alpha}))$ and so on, and let $u_a \equiv u(w-d)$ and $u_{na} \equiv u(w)$ denote the state contingent outside option utility. Moreover, I denote $v \equiv u^{-1}$ the inverse function of u. Since $u(w)$ is strictly increasing in w, this inverse function exists. Moreover, since $u(w)$ is strictly concave in w, v is strictly convex in its argument. Moreover, by definition $v(u(w_i)) = w_i$ for $i = a, na$.

In this notation I can write the insurance company's problem as follows:

$$\max_{\underline{u}_a, \underline{u}_{na}, \overline{u}_a, \overline{u}_{na}} \left\{ \begin{array}{c} \lambda\left(-\underline{\theta}v(\underline{u}_a) - (1-\underline{\theta}) v(\underline{u}_{na}) + w - \underline{\theta}d\right) \\ +(1-\lambda)\left(-\overline{\theta}v(\overline{u}_a) - \left(1-\overline{\theta}\right) v(\overline{u}_{na}) + w - \overline{\theta}d\right) \end{array} \right\} \tag{1}$$

$$s.t.$$

$$\overline{\theta}\overline{u}_a + \left(1-\overline{\theta}\right)\overline{u}_{na} \geq \overline{\theta}\underline{u}_a + \left(1-\overline{\theta}\right)\underline{u}_{na}, \tag{2}$$

$$\underline{\theta}\underline{u}_a + (1-\underline{\theta})\underline{u}_{na} \geq \underline{\theta}\overline{u}_a + (1-\underline{\theta})\overline{u}_{na}, \tag{3}$$

$$\overline{\theta}\overline{u}_a + \left(1-\overline{\theta}\right)\overline{u}_{na} \geq \overline{\theta}u_a + \left(1-\overline{\theta}\right)u_{na}, \text{ and} \tag{4}$$

$$\underline{\theta}\underline{u}_a + (1-\underline{\theta})\underline{u}_{na} \geq \underline{\theta}u_a + (1-\underline{\theta})u_{na} \tag{5}$$

I observe that problem (1) s.t. (2) – (5) has the same structure as the Maskin Riley problem has with the inessential difference that the firm's profit function depends on θ. For this reason I conjecture that I can apply the same technique to solve my problem. In particular, I will solve a "reduced problem", that I obtain from the full problem when I conjecture that constraints (2) and

(5) hold with equality and that the remaining constraints are slack. As is usual, I will show that the solution I obtain from the reduced problem satisfies the two neglected constraints (3) and (4). Therefore, the solution to the reduced problem coincides with the solution of the full problem.

Given that (5) holds with equality, I can write

$$\underline{u}_a = s(\underline{u}_{na}) \equiv u_a + \frac{(1-\underline{\theta})}{\underline{\theta}}(u_{na} - \underline{u}_{na}). \tag{6}$$

Likewise, imposing (2) with equality, I can write

$$\overline{u}_a = \underline{u}_a + \frac{(1-\overline{\theta})}{\overline{\theta}}(\underline{u}_{na} - \overline{u}_{na}).$$

Using (6) I can simplify this condition further and obtain

$$\overline{u}_a = t(\underline{u}_{na}, \overline{u}_{na}) \equiv u_a + \frac{(1-\underline{\theta})}{\underline{\theta}}(u_{na} - \underline{u}_{na}) + \frac{(1-\overline{\theta})}{\overline{\theta}}(\underline{u}_{na} - \overline{u}_{na}). \tag{7}$$

Substituting $s(\underline{u}_{na})$ for $\underline{u}_{na}$ and $t(\underline{u}_{na}, \overline{u}_{na})$ for $\overline{u}_a$ into (1) I can write the reduced problem as the following, unconstrained maximization problem:

$$\max_{\underline{u}_{na}, \overline{u}_{na}} \left\{ \begin{array}{c} \lambda\left(-\underline{\theta} v(s(\underline{u}_{na})) - (1-\underline{\theta}) v(\underline{u}_{na}) + w - \underline{\theta} d\right) \\ +(1-\lambda)\left(-\overline{\theta} v(t(\underline{u}_{na}, \overline{u}_{na})) - \left(1-\overline{\theta}\right) v(\overline{u}_{na}) + w - \overline{\theta} d\right) \end{array} \right\}. \tag{8}$$

Let $\{\underline{u}_{na}^*, \underline{u}_a^*, \overline{u}_{na}^*, \overline{u}_a^*\}$ denote a solution to the reduced problem.

Proposition 1 *For high enough λ the solution to the monopolist's problem is characterized by the conditions*

$$\overline{u}_{na}^* = \overline{u}_a^*, \tag{9}$$

$$\frac{1}{u'(v(\underline{u}_a^*))} = \frac{1}{u'(v(\underline{u}_{na}^*))} - \frac{(1-\lambda)}{\lambda} \frac{1}{u'(v(\overline{u}_a^*))} \frac{\overline{\theta} - \underline{\theta}}{\underline{\theta}(1-\underline{\theta})}, \tag{10}$$

(7), *and* (6).

Proof. Problem (8) is concave in the variables $\underline{u}_{na}$ and $\overline{u}_{na}$, by the fact that $v(\cdot)$ is strictly convex, which in turn is equivalent to $u(\cdot)$ being strictly concave. Therefore, the first-order conditions are necessary and sufficient for an optimum. Conditions (9) and (10) follow immediately from the first-order conditions, in which I replace $v'(\underline{u}_a)$ by $\frac{1}{u'(v(\underline{u}_a))}$ and so on. The result that for low enough λ low taste consumers do not participate is well known and can also be obtained directly from the comparative statics of the system of equations (9), (10), (7), and (6) with respect to λ. The remainder of the proof, showing that the solution corresponds to the solution of the full problem is in the appendix. ■

The solution displays the classical features of no distortion at the top, too little insurance for the low risk individuals, and no rent at the bottom. The contract offered to the low demand consumer determines the rent the monopolist has to leave to the high demand consumer. At the optimum the low demand consumer receives too little insurance relative to the social optimum. The reason is that this makes it possible for the monopolist to extract more rents from the high demand consumer.

It is straightforward to do comparative statics of the solution, notably with respect to the fraction of high risks. Suppose the problem is such that the monopolist sells insurance to both consumer types. I can reduce the system of equations to a single condition and apply the implicit function theorem to this equation to obtain comparative statics predictions. Details are in the appendix. I summarize my results in the following proposition.

Proposition 2 *Whenever both consumers are served, the equilibrium utilities of the consumer with the low probability of accident satisfy* $\frac{\partial \underline{u}_{na}^*}{\partial \lambda} < 0$ *and* $\frac{\partial \underline{u}_a^*}{\partial \lambda} > 0$. *The equilibrium utilities of the consumer with the high probability of accident satisfy* $\frac{\partial}{\partial \lambda}\overline{u}_{na}^* = \frac{\partial}{\partial \lambda}\overline{u}_a^* > 0$.

The higher the fraction of low risk consumers, the more insurance these consumers receive, that is $\frac{\partial \underline{u}_{na}^*}{\partial \lambda} - \frac{\partial \underline{u}_a^*}{\partial \lambda} < 0$. On the other hand, high risk consumers receive higher utilities the larger the fraction of low risk consumers. The rationale for these results is that the more low risk consumers there are, the less importance the monopolist attaches to extracting rents from high risk consumers.

The results I have established in this section are not surprising. However, what is surprising is how difficult it is to prove these results in a direct approach and how easy it is to prove them using my indirect approach. This added analytical ease makes it possible to solve the more complex case of a continuum of types, to which I now turn.

3 The Case of a Continuum of Types

Assume now that there is a continuum of types in the market with probability of accident $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$. Let θ be distributed with a differentiable density $f(\theta)$ and cdf $F(\theta)$. The monopolist's problem is

$$\max_{u_a(\cdot), u_{na}(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \{-\theta v(u_a(\theta)) - (1-\theta) v(u_{na}(\theta)) + w - \theta d\} f(\theta)\, d\theta \tag{11}$$

$$s.t.\ for\, all\, \theta$$

$$\theta u_a(\theta) + (1-\theta) u_{na}(\theta) \geq \theta u_a\left(\hat{\theta}\right) + (1-\theta) u_{na}\left(\hat{\theta}\right) \forall \hat{\theta} \text{ and} \tag{12}$$

$$\theta u_a(\theta) + (1-\theta) u_{na}(\theta) \geq \theta \underline{u}_a + (1-\theta) \underline{u}_{na}. \tag{13}$$

Constraint (12) states that an insurant of type θ should prefer the contract designed for type θ rather than any other contract. Constraint (13) states that all types should prefer to participate. Since the insurance company can always offer contracts that are equivalent to the null contract, this formulation of the participation constraint is without loss of generality. To solve this problem, I derive an equivalent description of the set of implementable contracts, i.e., of the contracts that satisfy constraints (12) and (13). As a preliminary step towards that end, it is useful to observe the following:

Lemma 1 *A pair of utility schedules that solves problem* (11) *subject to* (12) *and* (13) *satisfies*

$$u_{na}(\theta) - u_a(\theta) \leq u_{na} - u_a \text{ for all } \theta. \tag{14}$$

Proof. Suppose condition (14) is violated for some θ, so

$$u_{na}(\theta) - u_a(\theta) > u_{na} - u_a.$$

Rearranging, multiplying by θ and adding u_{na} on both sides, I get

$$u_{na} + \theta(u_a - u_{na}) > u_{na} + \theta(u_a(\theta) - u_{na}(\theta)).$$

By (13), it follows that $u_{na}(\theta) > u_{na}$ to make type θ willing to participate. This implies that $u_a(\theta) \leq u_a$; if it was the case that $u_a(\theta) > u_a$, then the firm would raise the insuree's utility in both states, so the firm would incur a loss from trading with this insuree. Hence, the firm would be better off offering the null contract u_a, u_{na} to type θ. Since adding the null contract to the menu of contract offers can always be done without violating incentive or participation constraints, any menu that includes a contract $u_a(\theta), u_{na}(\theta)$ for some θ where $u_{na}(\theta) > u_{na}$ and $u_a(\theta) > u_a$ cannot be optimal.

I now show that also contracts where $u_{na}(\theta) > u_{na}$ and $u_a(\theta) \leq u_a$ are loss-makers for the firm. To see this, consider a decision-maker with increasing and convex utility function evaluating lotteries $A \equiv \{u_a(\theta), u_{na}(\theta), \theta\}$ and $B \equiv \{u_a, u_{na}, \theta\}$. By (13) for type θ, lottery A has a weakly higher expected value than lottery B has. Since $u_{na}(\theta) > u_{na}$ and $u_a(\theta) \leq u_a$, lottery A has a wider support than lottery B. Since v is increasing and concave, it follows that

$$\theta v(u_a(\theta)) + (1-\theta) v(u_{na}(\theta)) \geq \theta v(u_a) + (1-\theta) v(u_{na}).$$

Since this inequality holds for any convex and increasing v, it holds in particular for $v = u^{-1}$. Rearranging the last inequality, and adding $w - \theta d$ on both sides of the inequality, I get

$$-\theta v(u_a(\theta)) - (1-\theta) v(u_{na}(\theta)) + w - \theta d \leq -\theta v(u_a) - (1-\theta) v(u_{na}) + w - \theta d = 0.$$

The last equality states simply that the firm receives zero profit from offering the null-contract to the insuree. Hence, any contract that has $u_{na}(\theta) > u_{na}$ and $u_a(\theta) \leq u_a$ must be a loss maker for the firm.

So we have shown that $u_{na}(\theta) - u_a(\theta) > u_{na} - u_a$ cannot hold for any θ, which proves (14) must hold. ■

The intuition for this result is pretty straightforward: in any optimal contract the insurance company reduces the risk the consumers face in the sense that the difference between utility levels with and without an accident are reduced relative to the consumers' autarky situations. If accident probabilities were known to the firm, condition (14) would be obvious. Lemma 1 demonstrates that the result carries over to the case of unkown accident probabilities. The Lemma is useful because condition (14) is needed to prove the following, powerful result:

Proposition 3 *A pair of utility schedules $u_a(\theta)$, $u_{na}(\theta)$ is implementable if and only if*

$$u_a'(\theta) \geq 0 \geq u_{na}'(\theta) \tag{15}$$

and in addition

$$u_{na}(\theta) = u_{na} + \frac{\underline{\theta}}{1-\underline{\theta}}u_a - \frac{\theta}{1-\theta}u_a(\theta) + \int_{\underline{\theta}}^{\theta} \frac{1}{(1-z)^2}u_a(z)\,dz. \tag{16}$$

The proof uses standard arguments. The crucial difference to the approach with a risk neutral agent is that the switch to maximization with respect to indirect utility instead of transfers is not useful here. But, using the standard arguments, I can eliminate one utility schedule, $u_{na}(\cdot)$, from the insurance company's problem. Finally, I can impose the individual rationality constraint at the low bound of the support, and this is sufficient to ensure the participation constraint is satisfied for all θ. The reason is that (14) ensures that the value of the inside option - the insuree's indirect utility from choosing optimally from the menu of contracts - is increasing at least as fast with θ than the outside option - the consumer's utility without insurance - does.

Define the auxiliary variables $y(\theta) \equiv \frac{u_a(\theta)}{(1-\theta)^2}$ and $x(\theta) \equiv \int_{\underline{\theta}}^{\theta} \frac{1}{(1-z)^2}u_a(z)\,dz$. These variables are constructed such that they satisfy $y(\theta) = x'(\theta)$. Using these auxiliary variables I can write condition (16) equivalently as

$$u_{na}(\theta) = \tau(y(\theta), x(\theta), \theta) \equiv u_{na} + \frac{\underline{\theta}}{1-\underline{\theta}}u_a - \theta(1-\theta)y(\theta) + x(\theta). \tag{17}$$

Substituting $y(\theta)(1-\theta)^2$ for $u_a(\theta)$ and $\tau(y(\theta), x(\theta), \theta)$ for $u_{na}(\theta)$ into (11) I obtain a problem that is equivalent to problem (11) subject to (12) and (13), but much easier to analyze:[3]

$$\max_y \int_{\underline{\theta}}^{\overline{\theta}} \left\{-\theta v\left((1-\theta)^2 y\right) - (1-\theta) v(\tau(y,x,\theta)) + w - \theta d\right\} f(\theta)\, d\theta \tag{18}$$

$$s.t.$$

$$\dot{x} = y\,; x(\underline{\theta}) = 0;\ x(\overline{\theta})\ free;\ \text{and} \tag{19}$$

$$-2y + (1-\theta)\dot{y} \geq 0. \tag{20}$$

Condition (20) is equivalent to and replaces the monotonicity condition (15). To solve my problem, I proceed as is usual. I impose sufficient conditions on the distribution of types that allow me to neglect the monotonicity constraint.

Proposition 4 *Suppose the density satisfies*

$$\frac{f'(\theta)}{f(\theta)} \geq \frac{1}{\theta}\frac{3\theta-2}{1-\theta} \forall \theta \in \left[\underline{\theta}, \overline{\theta}\right]. \tag{21}$$

Then, a set of insurance contracts is optimal if and only if the utility schedules satisfy

$$\frac{1}{u'(v(u_a(\theta)))} = \frac{1}{u'(v(u_{na}(\theta)))} - \frac{\int_{\theta}^{\overline{\theta}} (1-z) \frac{1}{u'(v(u_{na}(z)))} f(z)\, dz}{\theta(1-\theta)^2 f(\theta)} \tag{22}$$

and condition (16).

Proof. The reduced problem is an unconstrained problem of optimal control with one state and one control variable, and initial condition for the state variable. It is a fixed endpoint problem. The Hamiltonian for this problem is

$$H = \left\{-\theta v\left((1-\theta)^2 y\right) - (1-\theta) v(\tau(y,x,\theta)) + w - \theta d\right\} f(\theta) + \lambda y,$$

where λ is the costate variable. The Pontryagin (necessary) conditions for an optimal policy are

$$\frac{\partial H}{\partial y} = \left(-\theta(1-\theta)^2 v'\left((1-\theta)^2 y\right) + \theta(1-\theta)^2 v'(\tau(y,x,\theta))\right) f(\theta) + \lambda = 0, \tag{23}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = (1-\theta) v'(\tau(y,x,\theta)) f(\theta),\ \text{and} \tag{24}$$

$$\lambda(\overline{\theta}) = 0. \tag{25}$$

The last equality is the transversality condition.

[3] I follow the usual conventions of optimal control theory: I drop the dependence on θ where this can be done without causing confusion, and I switch to dot notation to denote derivates.

Using the transversality condition (25) and the equation of motion for the costate variable (24) I find

$$\lambda(\theta) = \lambda(\overline{\theta}) - \int_{\theta}^{\overline{\theta}} \dot{\lambda} d\tau = -\int_{\theta}^{\overline{\theta}} (1-\tau) v'(\tau(y,x,\tau)) f(\tau) d\tau.$$

Substituting for λ in condition (23) I obtain the condition

$$v'(u_a(\theta)) = v'(u_{na}(\theta)) - \frac{\int_{\theta}^{\overline{\theta}} (1-\tau) v'(u_{na}(\tau)) f(\tau) d\tau}{\theta(1-\theta)^2 f(\theta)}.$$

To obtain condition (22) I switch notation using $v'(u_a(\theta)) = \frac{1}{u'(v(u_a(\theta)))}$ and so on.

Finally, the individual rationality constraint at $\theta = \underline{\theta}$ completely pins down the path of utility schedules.

The proofs of sufficiency and monotonicity are in the appendix. ■

An efficient solution would equalize the consumer's marginal utilities across the two states. From condition (22) it is apparent that the consumer's marginal utility is larger if he has an accident, meaning that he receives less than full insurance. In order to decrease the rent of the consumers with a relatively high demand for insurance (those with a high θ), the monopolist makes it less attractive for these consumers to understate their demand for insurance. To this end the consumers with a relatively low demand for insurance (those with a low θ) receive too little insurance relative to the social optimum.

It is instructive to compare the structure of the solution with the structure of other problems of non-linear pricing. E.g., following Maskin and Riley (1984a) suppose buyers are risk neutral with a utility function $\theta V(q) - T(q)$ where q is the quantity of some good, $V(\cdot)$ is a concave function and $T(\cdot)$ a non-linear tariff. Suppose in addition the seller has constant marginal cost of production c. Then we can write the solution (provided that $\frac{(1-F(\theta))}{f(\theta)}$ is non-increasing to guarantee monotonicity) as $(\theta V'(q(\theta)) - c) f(\theta) = (1-F(\theta)) V'(q(\theta))$. The expected loss arising from the departure from first-best is equated to the marginal reduction in the information rent of all types with a larger marginal utility of consumption. (22) would take essentially the same form if the marginal utility of consumption were constant in case there is no accident. So the essential economic difference to the model of Maskin and Riley (1984a) is that the type θ impacts on the marginal utility of consumption in both states.

A further technical difference is that the set of distributions that generate monotonic solutions differs. One may wonder which distributions do satisfy (21). The following Lemma shows that condition (21) is satisfied for a relatively rich class of distributions.

Lemma 2 *Any log-concave density $f(\theta)$ that satisfies $\frac{f'(\overline{\theta})}{f(\overline{\theta})} \geq \frac{1}{\overline{\theta}} \frac{3\overline{\theta}-2}{1-\overline{\theta}}$, satisfies (21) for all $\theta \in$*

$\left[\underline{\theta}, \overline{\theta}\right]$.

Proof. $f(\theta)$ is log-concave if and only if $\ln f(\theta)$ is concave. Hence, for a log-concave density, $\frac{f'(\theta)}{f(\theta)}$ is non-increasing. On the other hand, $\frac{1}{\theta}\frac{3\theta-2}{1-\theta}$ is increasing in θ. Given $\frac{f'(\overline{\theta})}{f(\overline{\theta})} \geq \frac{1}{\overline{\theta}}\frac{3\overline{\theta}-2}{1-\overline{\theta}}$, (21) is satisfied for $\theta = \overline{\theta}$. Hence, (21) is satisfied everywhere. ■

The conditions in the lemma can be interpreted as a joint restriction on the class of densities and their support in the following sense. Consider the class of log-concave densities that satisfy $\frac{f'(\underline{\theta})}{f(\underline{\theta})} \geq \frac{1}{\underline{\theta}}\frac{3\underline{\theta}-2}{1-\underline{\theta}}$. If the distribution satisfies also $\frac{f'(\overline{\theta})}{f(\overline{\theta})} \geq \frac{1}{\overline{\theta}}\frac{3\overline{\theta}-2}{1-\overline{\theta}}$, then it satisfies (21) and we are done. However, suppose it does not satisfy $\frac{f'(\overline{\theta})}{f(\overline{\theta})} \geq \frac{1}{\overline{\theta}}\frac{3\overline{\theta}-2}{1-\overline{\theta}}$. Then, we can generate a new distribution by truncating the distibution at the right at some $\overline{\theta}^f$ defined by the condition $\frac{f'\left(\overline{\theta}^f\right)}{f\left(\overline{\theta}^f\right)} = \frac{1}{\overline{\theta}^f}\frac{3\overline{\theta}^f-2}{1-\overline{\theta}^f}$. Obviously the truncated distribution satisfies (21) for all $\theta \in \left[\underline{\theta}, \overline{\theta}^f\right]$. Hence, in this sense, condition (21) is a joint restriction to logconcave densities on a support with a low enough upper bound. The class of distributions that meet condition (21) seems reasonably large given the complexity of the screening problem. A simple example that satisfies the conditions in Lemma 2 is the uniform for the case where $\overline{\theta} \leq \frac{2}{3}$. For a general treatment of log-concave densities, see An (1998).

4 Closed Form Solutions and Comparative Statics

In the remainder of this article I provide some closed form solutions that are - to the best of my knowledge - not known in the literature.

Conditions (22) and (16) together form a system of two integral equations. An equivalent description of the optimal menu of insurance contracts is obtained by differentiating these two equations. The resulting expression is a second order differential equation, which is in general nonlinear. Since nothing is known about the existence of solutions to these type of equations, I abstain from a general treatment and directly discuss a case that can be solved.

Assume from now on that the von Neumann-Morgenstern utility function displays constant relative risk aversion, i.e.,

$$u(w) = C\frac{w}{1-a}^{1-a}$$

with coefficient of relative risk aversion $a = \frac{1}{2}$. This particular form is convenient because it renders $v(u)$ quadratic in u, which implies that the differential equation to be solved becomes linear. To completely pin down its solution one has to assume special functional forms for the density $f(\theta)$. The model is in fact flexible enough to allow this exercise to be carried through for different density functions. Some results are gathered in the following proposition.[4]

Proposition 5 *Suppose that the density is of the form* $f(\theta) = \frac{\alpha_0}{\theta(1-\theta)^2}$ *where* α_0 *is chosen such that* $F\left(\overline{\theta}\right) = 1$*. Then the solution takes the form*

$$u_a(\theta) = \frac{K}{\frac{\left(\ln \underline{\theta} - \ln \overline{\theta}\right)\underline{\theta}}{\underline{\theta}-1} - 1}\left(\theta - 1 - \ln\theta + \ln\overline{\theta}\right)$$

and

$$u_{na}(\theta) = \frac{K}{\frac{\left(\ln \underline{\theta} - \ln \overline{\theta}\right)\underline{\theta}}{\underline{\theta}-1} - 1}(\theta - 1)$$

where $K = u_{na} + \frac{\underline{\theta}}{1-\underline{\theta}} u_a$*.*

Consumers receive more insurance (in the sense of $u_{na}(\theta) - u_a(\theta)$ *being smaller) the larger is the monetary loss* d *and the smaller is their wealth* w*.*

Consumers fare better under the optimal contract the wealthier they are.

The same qualitative features obtain if the density is $f(\theta) = \frac{\alpha_1}{(1-\theta)^3}$ *where* α_1 *is chosen such that* $F\left(\overline{\theta}\right) = 1$*. In this case, the solution takes the form*

$$u_a(\theta) = K\frac{2A_1\theta + \theta^2 - 2(\ln\theta)\theta - 1 - 4\theta}{\theta(A_1A_2 - A_3)}$$

and

$$u_{na}(\theta) = (\theta + A_1 - 3 - \ln\theta)\frac{K}{A_1A_2 - A_3}$$

where A_1, A_2, A_3 *are constants that are determined in the appendix.*

The comparative statics are intuitive. The driving force behind these results is the assumption of constant relative risk aversion. The wealthier consumers are, the lower is their demand for insurance. Consequently, the rent the monopolist can extract from the consumers is the smaller the wealthier the consumers are. As a result the departure from the efficient full insurance solution is smaller for each type when all consumers are wealthier. These results are reversed for the size of the damage d.

5 Conclusion

This paper makes a technical contribution. It provides a method to solve a class of screening problems that feature risk aversion and uncertainty in the form of two outcome distributions.

[4] It is straightforward to check that the densities in the following proposition satisfy condition (21) for $\overline{\theta}$ low enough. Another tractable case is the uniform distribution. However, the solution for this case takes quite a complicated form and is therefore omitted.

Using this method, I derive a complete solution for a model of monopolistic insurance with a continuum of types. A closed form description of contracts and comparative statics results are feasible for special cases of the model when consumers have CRRA utilities.

6 Appendix

Proof of Proposition 1 (cont.). Observe that the variable $\overline{u}_{na}$ enters the profit function (8) only through the part of the profit that stems from the high risk consumers. The first-order condition with respect to $\overline{u}_{na}$ is

$$(1-\lambda)\left(-\overline{\theta}v'\left(t\left(\underline{u}_{na},\overline{u}_{na}^{*}\right)\right)\frac{\partial t\left(\underline{u}_{na},\overline{u}_{na}^{*}\right)}{\partial \overline{u}_{na}}-\left(1-\overline{\theta}\right)v'\left(\overline{u}_{na}^{*}\right)\right)=0.$$

Substituting $\frac{\partial t\left(\underline{u}_{na},\overline{u}_{na}^{*}\right)}{\partial \overline{u}_{na}}=-\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}$ from (7) and simplifying, I get

$$\overline{\theta}v'\left(t\left(\underline{u}_{na},\overline{u}_{na}^{*}\right)\right)\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}-\left(1-\overline{\theta}\right)v'\left(\overline{u}_{na}^{*}\right)=0,$$

and hence (9). The first-order condition with respect to is

$$\lambda\left(-\underline{\theta}v'\left(s\left(\underline{u}_{na}^{*}\right)\right)\frac{\partial s\left(\underline{u}_{na}^{*}\right)}{\partial \underline{u}_{na}}-\left(1-\underline{\theta}\right)v'\left(\underline{u}_{na}^{*}\right)\right)+(1-\lambda)\left(-\overline{\theta}v'\left(t\left(\underline{u}_{na}^{*},\overline{u}_{na}\right)\right)\frac{\partial t\left(\underline{u}_{na}^{*},\overline{u}_{na}\right)}{\partial \underline{u}_{na}}\right)=0.$$

Substituting for $\frac{\partial s\left(\underline{u}_{na}\right)}{\partial \underline{u}_{na}}=-\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}$ from (6) and for $\frac{\partial t\left(\underline{u}_{na},\overline{u}_{na}\right)}{\partial \underline{u}_{na}}=-\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\overline{\theta}\underline{\theta}}$ from (7) and simplifying I get

$$\left(v'\left(s\left(\underline{u}_{na}^{*}\right)\right)-v'\left(\underline{u}_{na}^{*}\right)\right)+\frac{(1-\lambda)}{\lambda}v'\left(t\left(\underline{u}_{na}^{*},\overline{u}_{na}\right)\right)\frac{\left(\overline{\theta}-\underline{\theta}\right)}{\underline{\theta}\left(1-\underline{\theta}\right)}=0.$$

Substituting for $v'=\frac{1}{u'}$ and making use of (6) and (7) I get condition (10).

It remains to be shown that the pair of contracts characterized by the conditions in proposition 1 also satisfy the constraints (3) and (4). In fact, the high risk type is willing to participate, because

$$\begin{aligned}
\overline{\theta}\overline{u}_a+\left(1-\overline{\theta}\right)\overline{u}_{na} &= \overline{\theta}\underline{u}_a+\left(1-\overline{\theta}\right)\underline{u}_{na}\\
&= \left(\overline{\theta}-\underline{\theta}\right)\left(\underline{u}_a-\underline{u}_{na}\right)+\underline{\theta}\underline{u}_a+\left(1-\underline{\theta}\right)\underline{u}_{na}\\
&= \left(\overline{\theta}-\underline{\theta}\right)\left(\underline{u}_a-\underline{u}_{na}\right)+\underline{\theta}u_a+\left(1-\underline{\theta}\right)u_{na}\\
&= \left(\overline{\theta}-\underline{\theta}\right)\left(\underline{u}_a-\underline{u}_{na}+u_{na}-u_a\right)+\overline{\theta}u_a+\left(1-\overline{\theta}\right)u_{na}\\
&\geq \overline{\theta}u_a+\left(1-\overline{\theta}\right)u_{na}.
\end{aligned}$$

The first equality uses the binding constraint (7), the second is simple algebra, the third uses the binding constraint (6), and the fourth equality follows again by simple algebra. Finally, I prove in Lemma 1 that $\underline{u}_{na}-\underline{u}_a\leq u_{na}-u_a$, which implies the last inequality.

Since we impose (6), the constraint (3) is equivalent to

$$\underline{\theta}\left(u_a-\overline{u}_a\right)+\left(1-\underline{\theta}\right)\left(u_{na}-\overline{u}_{na}\right)>0.$$

Substituting $\overline{u}_a$ from (7) this inequality is seen to be equivalent to the condition

$$\underline{\theta}\left(-\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\left(u_{na}-\underline{u}_{na}\right)-\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}\left(\underline{u}_{na}-\overline{u}_{na}\right)\right)+\left(1-\underline{\theta}\right)\left(u_{na}-\overline{u}_{na}\right)>0.$$

Simplifying I obtain the condition

$$\left(\frac{\overline{\theta}-\underline{\theta}}{\overline{\theta}}\right)\left(\underline{u}_{na}-\overline{u}_{na}\right)>0.$$

Economically, the low types are willing to buy the bundle intended for them when insurance is less costly to low risks than to high risks. ■

Proof of Proposition 2. Substituting (9) into (7) I obtain

$$\overline{u}^*_{na}=u_a+\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\left(u_{na}-\underline{u}^*_{na}\right)+\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}\left(\underline{u}^*_{na}-\overline{u}^*_{na}\right),$$

which can be solved for $\overline{u}^*_{na}$. I obtain this equation for

$$\overline{u}^*_{na}=T\left(\underline{u}^*_{na}\right)\equiv\overline{\theta}\left(u_a+\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}u_{na}+\left(\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}-\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\right)\left(\underline{u}^*_{na}\right)\right). \tag{26}$$

Condition (26) allows me to write

$$t\left(\underline{u}^*_{na},\overline{u}^*_{na}\right)=t\left(\underline{u}^*_{na},T\left(\underline{u}^*_{na}\right)\right)=\overline{\theta}u_a+\overline{\theta}\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}u_{na}+\left(\left(1-\overline{\theta}\right)-\frac{\overline{\theta}\left(1-\underline{\theta}\right)}{\underline{\theta}}\right)\underline{u}^*_{na}. \tag{27}$$

From condition (6) I have

$$\underline{u}^*_{a}=s\left(\underline{u}^*_{na}\right)=u_a+\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\left(u_{na}-\underline{u}^*_{na}\right). \tag{28}$$

Now I can use conditions (27) and (28) to write condition (10) in more compact notation. I obtain the condition

$$v'\left(s\left(\underline{u}^*_{na}\right)\right)-v'\left(\underline{u}^*_{na}\right)+\frac{\left(1-\lambda\right)}{\lambda}v'\left(t\left(\underline{u}^*_{na},T\left(\underline{u}^*_{na}\right)\right)\right)\frac{\overline{\theta}-\underline{\theta}}{\underline{\theta}\left(1-\underline{\theta}\right)}=0. \tag{29}$$

Define

$$\Omega\left(\underline{u}^*_{na},\zeta\right)\equiv v'\left(s\left(\underline{u}^*_{na}\right)\right)-v'\left(\underline{u}^*_{na}\right)+\frac{\left(1-\lambda\right)}{\lambda}v'\left(t\left(\underline{u}^*_{na},T\left(\underline{u}^*_{na}\right)\right)\right)\frac{\overline{\theta}-\underline{\theta}}{\underline{\theta}\left(1-\underline{\theta}\right)}.$$

where ζ is a parameter of the problem. By the implicit function theorem,

$$\frac{\partial\underline{u}^*_{na}}{\partial\zeta}=\frac{\frac{\partial\Omega\left(\underline{u}^*_{na},\zeta\right)}{\partial\zeta}}{-\frac{\partial\Omega\left(\underline{u}^*_{na},\zeta\right)}{\partial\underline{u}^*_{na}}}.$$

$$\begin{aligned}\frac{\partial \Omega\left(\underline{u}_{na}^{*},\zeta\right)}{\partial \underline{u}_{na}^{*}} &= v''\left(s\left(\underline{u}_{na}^{*}\right)\right)\frac{\partial s\left(\underline{u}_{na}^{*}\right)}{\partial \underline{u}_{na}^{*}} - v''\left(\underline{u}_{na}^{*}\right) \\ &+\frac{(1-\lambda)}{\lambda}v''\left(t\left(\underline{u}_{na}^{*},T\left(\underline{u}_{na}^{*}\right)\right)\right)\frac{\overline{\theta}-\underline{\theta}}{\underline{\theta}\left(1-\underline{\theta}\right)}\left(\frac{\partial t\left(\underline{u}_{na}^{*},T\left(\underline{u}_{na}^{*}\right)\right)}{\partial \underline{u}_{na}^{*}}+\frac{\partial t\left(\underline{u}_{na}^{*},T\left(\underline{u}_{na}^{*}\right)\right)}{\partial \tau}\frac{\partial T\left(\underline{u}_{na}^{*}\right)}{\partial \underline{u}_{na}^{*}}\right).\end{aligned}$$

Simplification gives

$$\begin{aligned}\frac{\partial \Omega\left(\underline{u}_{na}^{*},\zeta\right)}{\partial \underline{u}_{na}^{*}} &= -v''\left(s\left(\underline{u}_{na}^{*}\right)\right)\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}} - v''\left(\underline{u}_{na}^{*}\right) \\ &+\frac{(1-\lambda)}{\lambda}v''\left(t\left(\underline{u}_{na}^{*},T\left(\underline{u}_{na}^{*}\right)\right)\right)\frac{\overline{\theta}-\underline{\theta}}{\underline{\theta}\left(1-\underline{\theta}\right)}\left(\left(1-\overline{\theta}\right)-\frac{\overline{\theta}\left(1-\underline{\theta}\right)}{\underline{\theta}}\right)<0.\end{aligned}$$

where the inequality follows from v being strictly convex and $\left(1-\overline{\theta}\right)-\frac{\overline{\theta}(1-\underline{\theta})}{\underline{\theta}}$ being negative.

Consider now $\frac{\partial\Omega\left(\underline{u}_{na}^{*},\zeta\right)}{\partial\zeta}$. For the case $\zeta=\lambda$ I obtain

$$\frac{\partial \Omega\left(\underline{u}_{na}^{*},\lambda\right)}{\partial \lambda} = -\frac{1}{\lambda^{2}}v'\left(t\left(\underline{u}_{na}^{*},T\left(\underline{u}_{na}^{*}\right)\right)\right)\frac{\overline{\theta}-\underline{\theta}}{\underline{\theta}\left(1-\underline{\theta}\right)}<0$$

and thus

$$\frac{d\underline{u}_{na}^{*}}{d\lambda}<0.$$

From (28) I have then that

$$\frac{d\underline{u}_{a}^{*}}{d\lambda}=\frac{\partial s\left(\underline{u}_{na}^{*}\right)}{\partial \underline{u}_{na}^{*}}\frac{d\underline{u}_{na}^{*}}{d\lambda}=-\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\frac{d\underline{u}_{na}^{*}}{d\lambda}>0.$$

Finally, from (26) , I get

$$\frac{d\overline{u}_{a}^{*}}{d\lambda}=\frac{d\overline{u}_{na}^{*}}{d\lambda}=\overline{\theta}\left(\frac{\left(1-\overline{\theta}\right)}{\overline{\theta}}-\frac{\left(1-\underline{\theta}\right)}{\underline{\theta}}\right)\frac{d\underline{u}_{na}^{*}}{d\lambda}>0.$$

■

Proof of Proposition 3. A necessary condition for an optimal report is the first-order condition

$$\theta u_{a}'\left(\hat{\theta}\right)+(1-\theta)\,u_{na}'\left(\hat{\theta}\right)\Big|_{\hat{\theta}=\theta}=0. \tag{30}$$

Notice that $u_{a}'\left(\hat{\theta}\right)$ and $u_{na}'\left(\hat{\theta}\right)$ must have opposing signs. A total differentiation of condition (30) gives

$$\left(\theta u_{a}''(\theta)+(1-\theta)\,u_{na}''(\theta)\right)+\left(u_{a}'(\theta)-u_{na}'(\theta)\right)=0.$$

Thus, truth-telling constitutes a locally optimal strategy only if $u_{a}'(\theta)-u_{na}'(\theta)\geq 0$. In combination with the observation that $u_{a}'(\theta)$ and $u_{na}'(\theta)$ cannot have the same sign, the monotonicity condition follows.

Let $U(\theta)=\max_{\hat{\theta}}\theta u_{a}\left(\hat{\theta}\right)+(1-\theta)\,u_{na}\left(\hat{\theta}\right)$. By the envelope theorem

$$U'(\theta)=u_{a}(\theta)-u_{na}(\theta).$$

On the other hand, the change in the outside option when the type changes is

$$\frac{\partial}{\partial\theta}\left(\theta u_a + (1-\theta)\, u_{na}\right) = u_a - u_{na}.$$

Hence, whenever

$$u_a - u_{na} \leq u_a(\theta) - u_{na}(\theta)$$

then the value of the outside option decreases faster with θ than the value of the inside option. Consequently, it is sufficient to impose the individual rationality constraint at the lower bound of the support. Note that the individual rationality constraint must bind somewhere. Otherwise the insurer could lower all utility levels by the same amount and increase his profit.

To get the integral condition, observe that by definition

$$u_{na}(\theta) = u_{na}(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u'_{na}(z)\, dz. \tag{31}$$

From the first-order condition,

$$u'_{na}(\theta) = -\frac{\theta}{(1-\theta)} u'_a(\theta).$$

From the individual rationality condition at $\theta = \underline{\theta}$ I have

$$u_{na}(\underline{\theta}) = \frac{\underline{\theta}}{(1-\underline{\theta})} u_a - \frac{\underline{\theta}}{(1-\underline{\theta})} u_a(\underline{\theta}) + u_{na}.$$

Substitution of these two conditions into (31) gives

$$\begin{aligned} u_{na}(\theta) &= u_{na} + \frac{\underline{\theta}}{(1-\underline{\theta})} u_a - \frac{\underline{\theta}}{(1-\underline{\theta})} u_a(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \frac{z}{1-z} u'_a(z)\, dz \\ &= u_{na} + \frac{\underline{\theta}}{(1-\underline{\theta})} u_a - \frac{\underline{\theta}}{(1-\underline{\theta})} u_a(\underline{\theta}) - \left(\frac{\theta}{1-\theta} u_a(\theta) - \frac{\underline{\theta}}{(1-\underline{\theta})} u_a(\underline{\theta})\right) \\ &\quad + \int_{\underline{\theta}}^{\theta} \frac{1}{(1-z)^2} u_a(z)\, dz. \end{aligned}$$

Finally, the monotonicity condition makes the local conditions sufficient for the global maximization conditions. I can write

$$\begin{aligned} \theta u_a(\theta) + (1-\theta)\, u_{na}(\theta) &\geq \left(\hat{\theta} + \theta - \hat{\theta}\right) u_a\left(\hat{\theta}\right) + \left((1-\theta) - \left(1-\hat{\theta}\right) + \left(1-\hat{\theta}\right)\right) u_{na}\left(\hat{\theta}\right), \text{ or} \\ U(\theta) &\geq U\left(\hat{\theta}\right) + \left(\theta - \hat{\theta}\right)\left(u_a\left(\hat{\theta}\right) - u_{na}\left(\hat{\theta}\right)\right). \end{aligned}$$

Since $U'(\theta) = u_a(\theta) - u_{na}(\theta)$ I can write

$$U\left(\hat{\theta}\right) = U(\theta) + \int_{\theta}^{\hat{\theta}} \left(u_a(z) - u_{na}(z)\right) dz.$$

Hence, global incentive compatibility requires that

$$\begin{aligned}
0 &\geq \int_{\theta}^{\hat{\theta}} (u_a(z) - u_{na}(z))\, dz + \left(\theta - \hat{\theta}\right)\left(u_a\left(\hat{\theta}\right) - u_{na}\left(\hat{\theta}\right)\right) \\
&= \int_{\theta}^{\hat{\theta}} (u_a(z) - u_{na}(z))\, dz - \int_{\theta}^{\hat{\theta}} \left(u_a\left(\hat{\theta}\right) - u_{na}\left(\hat{\theta}\right)\right) dz \\
&= \int_{\theta}^{\hat{\theta}} \left(\left(u_a(z) - u_a\left(\hat{\theta}\right)\right) + \left(u_{na}\left(\hat{\theta}\right) - u_{na}(z)\right)\right) dz.
\end{aligned}$$

I can now use the monotonicity condition $u_a'(\theta) \geq 0 \geq u_{na}'(\theta)$ to show that this inequality is satisfied. There are two cases to distinguish, $\hat{\theta} > \theta$ and the reverse. For $\hat{\theta} > \theta$ both the $\left(u_a(z) - u_a\left(\hat{\theta}\right)\right)$ and the term $\left(u_{na}\left(\hat{\theta}\right) - u_{na}(z)\right)$ are pointwise non-positive and the result follows directly. The case $\hat{\theta} < \theta$ is analogous and a discussion is omitted. ■

Proof of Proposition 4 (cont.). To prove sufficiency of the Pontryagin conditions I show that Mangasarian's sufficiency theorem applies. Specifically, since I have $\dot{x} = y$, a linear function, the theorem states that the first order condition is also sufficient for a maximum if H is concave in y and r jointly. Differentiating H twice with respect to y and/or x, respectively, I find

$$\frac{\partial^2 H}{\partial y^2} = \left(-\theta (1-\theta)^4 v''\left((1-\theta)^2 y\right) - \theta^2 (1-\theta)^3 v''(\tau(y,x,\theta))\right) f(\theta) < 0, \tag{32}$$

$$\frac{\partial^2 H}{\partial x^2} = -(1-\theta) v''(\tau(y,x,\theta)) f(\theta) < 0, \tag{33}$$

and

$$\frac{\partial^2 H}{\partial y \partial x} = \theta (1-\theta)^2 v''(\tau(y,x,\theta)) f(\theta) > 0. \tag{34}$$

Observe that $\frac{\partial^2 H}{\partial y \partial x} = -\theta(1-\theta)\frac{\partial^2 H}{\partial x^2}$. H is concave in x and y jointly if

$$\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial x^2} - \left(\frac{\partial^2 H}{\partial y \partial x}\right)^2 \geq 0.$$

Using the observation that $\frac{\partial^2 H}{\partial y \partial x} = -\theta(1-\theta)\frac{\partial^2 H}{\partial x^2}$, H is concave in x and y jointly if

$$\frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial x^2} \geq \theta^2 (1-\theta)^2 \left(\frac{\partial^2 H}{\partial x^2}\right)^2.$$

Dividing by $\frac{\partial^2 H}{\partial x^2} < 0$ and rearranging we have that H is concave if and only if, this condition is equivalent to

$$\frac{\partial^2 H}{\partial y^2} - \theta^2 (1-\theta)^2 \frac{\partial^2 H}{\partial x^2} \leq 0.$$

Substituting from (33) and (32) we find that

$$\frac{\partial^2 H}{\partial y^2} - \theta^2 (1-\theta)^2 \frac{\partial^2 H}{\partial x^2} = -\left(\theta (1-\theta)^4 v''\left((1-\theta)^2 y\right)\right) f(\theta) \leq 0.$$

Consider now the monotonicity condition. Taking derivatives in the condition of optimality

$$v'\left(u_a\left(\theta\right)\right)u_a'\left(\theta\right)-v'\left(u_{na}\left(\theta\right)\right)u_{na}'\left(\theta\right)=-\frac{\partial}{\partial\theta}\left(\frac{\int_\theta^{\overline{\theta}}\left(1-z\right)v'\left(u_{na}\left(z\right)\right)f\left(z\right)dz}{\theta\left(1-\theta\right)^2 f\left(\theta\right)}\right).$$

It follows that the monotonicity condition (15) is satisfied if

$$\frac{\partial}{\partial\theta}\left(\frac{\int_\theta^{\overline{\theta}}\left(1-z\right)v'\left(u_{na}\left(z\right)\right)f\left(z\right)dz}{\theta\left(1-\theta\right)^2 f\left(\theta\right)}\right)\leq 0.$$

or more explicitly if

$$\frac{\left(1-\theta\right)v'\left(u_{na}\left(\theta\right)\right)f\left(\theta\right)\left(\theta\left(1-\theta\right)^2 f\left(\theta\right)\right)+\frac{\partial}{\partial\theta}\left[\theta\left(1-\theta\right)^2 f\left(\theta\right)\right]\int_\theta^{\overline{\theta}}\left(1-\tau\right)v'\left(u_{na}\left(\tau\right)\right)f\left(\tau\right)d\tau}{\left(\theta\left(1-\theta\right)^2 f\left(\theta\right)\right)^2}\geq 0.$$

Observe that this inequality is satisfied if $\frac{\partial}{\partial\theta}\left[\theta\left(1-\theta\right)^2 f\left(\theta\right)\right]\geq 0$. The more interesting case is when $\frac{\partial}{\partial\theta}\left[\theta\left(1-\theta\right)^2 f\left(\theta\right)\right]$ can be negative. In this case, use the first-order condition for an optimal policy and rearrange it to conclude that

$$\begin{aligned}v'\left(u_{na}\left(\theta\right)\right)\theta\left(1-\theta\right)^2 f\left(\theta\right) &= v'\left(u_a\left(\theta\right)\right)\theta\left(1-\theta\right)^2 f\left(\theta\right)+\int_\theta^{\overline{\theta}}\left(1-z\right)v'\left(u_{na}\left(z\right)\right)f\left(z\right)dz\\ &> \int_\theta^{\overline{\theta}}\left(1-z\right)v'\left(u_{na}\left(z\right)\right)f\left(z\right)dz.\end{aligned}$$

Hence, I can substitute $\int_\theta^{\overline{\theta}}\left(1-z\right)v'\left(u_{na}\left(z\right)\right)f\left(z\right)dz$ for $v'\left(u_{na}\left(\theta\right)\right)\theta\left(1-\theta\right)^2 f\left(\theta\right)$ in the above inequality. After this substitution, I find that a sufficient condition for monotonicity is that $\left(1-\theta\right)f\left(\theta\right)+\frac{\partial}{\partial\theta}\left[\theta\left(1-\theta\right)^2 f\left(\theta\right)\right]\geq 0$ or equivalently

$$\frac{f'\left(\theta\right)}{f\left(\theta\right)}\geq-\frac{2-5\theta+3\theta^2}{\theta\left(1-\theta\right)^2}=\frac{1}{\theta}\frac{3\theta-2}{1-\theta}.$$

■

Proof of Proposition 5. Note that for the square root utility function we have $v\left(u\right)=\left(\frac{u}{2C}\right)^2$, $v'\left(u\right)=\frac{u}{2C^2}$, and $v''\left(u\right)=\frac{1}{2C^2}$, a constant.

Assume first that $f\left(\theta\right)=\frac{\alpha_0}{\theta\left(1-\theta\right)^2}$.

Differentiate condition (23) with respect to θ to obtain (using $f\left(\theta\right)=\frac{\alpha}{\theta\left(1-\theta\right)^2}$)

$$v''\left(\left(1-\theta\right)^2 y\right)\left(-2\left(1-\theta\right)y+\left(1-\theta\right)^2 z\right)-v''\left(\tau\left(y,x,\theta\right)\right)\left(2\theta y-\theta\left(1-\theta\right)z\right)=\frac{\dot{\lambda}}{\alpha_0},\qquad(35)$$

where $z=\dot{y}=\ddot{x}$. Since $v''\left(u\right)=\frac{1}{2C^2}$, a constant, we can simplify (35) to

$$\frac{1}{2C^2}\left(-2y+\left(1-\theta\right)z\right)=\frac{\dot{\lambda}}{\alpha_0}.$$

From (24) we have on the other hand

$$\frac{\dot{\lambda}}{\alpha_0} = \frac{1}{\theta(1-\theta)} \frac{1}{2C^2} (K - \theta(1-\theta) y + x) \tag{36}$$

where $K = u_{na} + \frac{\underline{\theta}}{1-\underline{\theta}} u_a$.

Putting equations (35) and (36) together we obtain the following second order linear differential equation:

$$(-2y + (1-\theta) z) = \frac{1}{\theta(1-\theta)} (K - \theta(1-\theta) y + x).$$

The equation has a solution[5] of the form

$$x(\theta) = \frac{K}{\theta - 1} + C_1 \frac{\theta}{\theta - 1} + C_2 \frac{1 + (\ln\theta)\theta}{\theta - 1}. \tag{37}$$

To determine the constant factors we use the boundary conditions. Condition (23) together with the transversality condition (25) give rise to the condition

$$\left(K + x\left(\overline{\theta}\right)\right) = \left(1 - \overline{\theta}\right) \dot{x}\left(\overline{\theta}\right).$$

Differentiating (37) and substituting the resulting expression for $\dot{x}(\theta)$ into this upper boundary condition gives

$$C_1 = -K - C_2\left(1 + \ln\overline{\theta}\right).$$

Substituting back into (37) gives

$$x(\theta) = -K + C_2 \frac{\left(\ln\theta - \ln\overline{\theta}\right)\theta}{\theta - 1} - C_2.$$

The second boundary condition is

$$x(\underline{\theta}) = -K + C_2 \frac{\left(\ln\underline{\theta} - \ln\overline{\theta}\right)\underline{\theta}}{\underline{\theta} - 1} - C_2 = 0.$$

Therefore,

$$C_2 = \frac{K}{\frac{\left(\ln\underline{\theta} - \ln\overline{\theta}\right)\underline{\theta}}{\underline{\theta} - 1} - 1}$$

and the final solution is

$$x(\theta) = -K + \frac{K}{\frac{\left(\ln\underline{\theta} - \ln\overline{\theta}\right)\underline{\theta}}{\underline{\theta} - 1} - 1} \left(\frac{\left(\ln\theta - \ln\overline{\theta}\right)\theta}{\theta - 1} - 1\right).$$

Recalling that $u_a(\theta) = (1-\theta)^2 \dot{x}(\theta)$ and $u_{na}(\theta) = K - \theta(1-\theta)\dot{x}(\theta) + x(\theta)$ one gets the expressions in the proposition.

[5] The solutions of differential equations in this proof were found using Maple software.

Assume now that $f(\theta) = \frac{\alpha_1}{(1-\theta)^3}$.

Differentiating again (23) and substituting from (24) we have

$$\frac{\partial}{\partial\theta}\left(\theta(1-\theta)^2 f(\theta)\right)\left(-v'\left((1-\theta)^2 y\right) + v'(\tau(y,x,\theta))\right) + (1-\theta)v'(\tau(y,x,\theta))f(\theta)$$
$$+\theta(1-\theta)^2 f(\theta)\left(\begin{array}{c} -v''\left((1-\theta)^2 y\right)\left(-2(1-\theta)y + (1-\theta)^2\dot{y}\right) \\ +v''(\tau(y,x,\theta))(2\theta y - \theta(1-\theta)\dot{y}) \end{array}\right) = 0.$$

The assumption implies that $\frac{\partial}{\partial\theta}\left(\theta(1-\theta)^2 f(\theta)\right) = (1-\theta)f(\theta)$. Moreover, $\frac{\partial}{\partial\theta}\left(\theta(1-\theta)^2 f(\theta)\right) = \frac{\alpha_1}{(1-\theta)^2}$. We can thus write

$$\left(-v'\left((1-\theta)^2 y\right) + 2v'(\tau(y,x,\theta))\right)$$
$$+\theta(1-\theta)\left(\begin{array}{c} -v''\left((1-\theta)^2 y\right)\left(-2(1-\theta)y + (1-\theta)^2\dot{y}\right) \\ +v''(\tau(y,x,\theta))(2\theta y - \theta(1-\theta)\dot{y}) \end{array}\right) = 0.$$

Simplification gives

$$-(1-\theta)^2 y + 2K + 2x - \theta(1-\theta)^2\dot{y} = 0.$$

The equation has a solution of the form

$$x(\theta) = -2\frac{K}{(1-\theta)} - \frac{C_1}{(1-\theta)}(\theta+1) - C_2\frac{(\ln\theta)\theta + \ln\theta + 4}{(1-\theta)}.$$

Differentiating we obtain

$$\dot{x}(\theta) = \frac{-2K\theta - 2C_1\theta + C_2\theta^2 - 2C_2(\ln\theta)\theta - C_2 - 4C_2\theta}{(1-\theta)^2\theta}.$$

To determine the constants of integration we use the boundary conditions. From the transversality condition (25) we obtain

$$C_2\frac{\left(1+\overline{\theta}+\left(\ln\overline{\theta}\right)\overline{\theta}\right)}{\overline{\theta}} + K = -C_1.$$

Substituting back into the solution and using the second boundary condition $x(\underline{\theta}) = 0$ we obtain

$$C_2 = \frac{K}{\frac{\left(1+\overline{\theta}+\left(\ln\overline{\theta}\right)\overline{\theta}\right)}{\overline{\theta}}\frac{(\underline{\theta}+1)}{(1-\underline{\theta})} - \frac{(\ln\underline{\theta})\underline{\theta}+\ln\underline{\theta}+4}{(1-\underline{\theta})}}.$$

Consequently

$$x(\theta) = -K + K\frac{A_1\frac{(\theta+1)}{(1-\theta)} - \frac{(\ln\theta)\theta+\ln\theta+4}{(1-\theta)}}{A_1A_2 - A_3},$$

where $\frac{\left(1+\overline{\theta}+\left(\ln\overline{\theta}\right)\overline{\theta}\right)}{\overline{\theta}} = A_1$, $\frac{(\underline{\theta}+1)}{(1-\underline{\theta})} = A_2$, and $\frac{(\ln\underline{\theta})\underline{\theta}+\ln\underline{\theta}+4}{(1-\underline{\theta})} = A_3$ is the final solution. Noting that

$$\dot{x}(\theta) = K\frac{2A_1\theta + \theta^2 - 2(\ln\theta)\theta - 1 - 4\theta}{(1-\theta)^2\theta(A_1A_2 - A_3)}$$

and recalling that $u_a(\theta) = (1-\theta)^2\dot{x}(\theta)$ and $u_{na}(\theta) = K - \theta(1-\theta)\dot{x}(\theta) + x(\theta)$ we obtain the expressions in the proposition. ■

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